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NUMERICAL ANALYSIS OF THE BRANCHED FORMS OF BENDING FOR A ROD

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Nonlinear boundary-value problems of plane bending of elastic arches subjected to uniformly distributed loading are solved numerically by the shooting method. The problems are formulated for a system of sixth-order ordinary differential equations that are more general than the Euler equations. Four variants of rod loading by transverse and longitudinal forces are considered. Branching of the solutions of boundary-value problems and the existence of intersected and isolated branches are shown. In the case of a translational longitudinal force, the classical Euler elasticas are obtained. The existence of a unique (rectilinear) form of equilibrium upon compression of a rod by a following longitudinal force is shown.

The nonlinear problems of plane bending of rods were analyzed mathematically by Popov [1, 2]. The analysis was based on the exact solution of one-dimensional quasilinear Euler equations in elliptic Legendre integrals. For a straight rod deformed by lateral loads, the possible forms of nonlinear bending were classified and the regions of their existence were determined. In [2], an algorithm for numerical solution of corresponding differential problems with the use of the difference approximation and Newton's iterative procedure was also proposed.

In the present work, the boundary-value problems of plane bending of a straight rod are formulated on the basis of more general equations that take into account the linear dependence between bending, tensile, and shear deformations [3]. The solution of these generalized equations is not expressed in terms of elliptic integrals. An effective algorithm for numerical analysis of nonlinear boundary-value problems by the shooting method is proposed.

We introduce the Cartesian coordinate system x_j with the orthonormalized base $\boldsymbol{e}_j$. The base line of the straight rod is set by the parametric equations

$$x_1 \equiv x_2 \equiv 0, \qquad x_3 = t \qquad \forall\ t \in [0, l], \tag{1}$$

where t is the internal parameter of the line and l is the initial length of the rod. According to (1), the base line in the initial state is a segment of the x_3 axis. We consider that the cross-sectional (profile) area A of the rod is constant, the base line passes through the geometrical center of the cross section, and the coordinate planes coincide with the planes of geometrical and material symmetry of the rod.

We consider the deformation of the rod caused by the force applied to the end ($t = l$) and specified by the vector

$$\boldsymbol{P} = P_2\boldsymbol{e}_2 + P_3\boldsymbol{e}_3. \tag{2}$$

The end $t = 0$ is rigidly fixed, i.e., its displacements and rotations are excluded. Under these conditions, a deformation of the rod under which the base line remains plane can occur:

$$x_1 \equiv 0, \qquad x_2 = y(t), \qquad x_3 = z(t) \qquad \forall\ t \in [0, l].$$

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Here y and z are the instantaneous coordinates of the point t in the plane (x_2, x_3). Deformation of this kind is generally called plane bending.

To analyze the plane bending of the rod subjected to a quasistatic load of the form (2), the nonlinear equations of the one-dimensional deformation model for a rod [3] are used. The goal of our analysis is to study the nonlinear kinematic effects. Therefore, the rod material is considered linearly-elastic and unboundedly strong.

Within the framework of the adopted restrictions, the elastic resistance of the rod is determined by the stiffness matrix

$$D = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix},$$

where G_1 and G_2 are the moduli of transverse shear and E_3 is the modulus of longitudinal tension or compression. In particular, for an isotropic material, we have $G_2 = G_1 = E_3/(2 + 2\nu)$, where ν is Poisson's ratio. The nonzero generalized stiffness matrices of the rod are calculated from the formulas of the form (2.7) given in [3]:

$$E = AD, \qquad E_{ii} = I_i D, \qquad I_i = \int\limits_A (x_i)^2 dA \quad (i = 1, 2).$$

The system of equations of nonlinear plane bending of a rod includes the generalized constitutive relations

$$X_{32} = AG_2U_{32}, \qquad X_{33} = AE_3U_{33}, \qquad Y = H\theta', \qquad H = I_2E_3, \tag{3}$$

the static equations following from [3, Eq. (1.7)]

$$X_{32} = P_2 \cos\theta + P_3 \sin\theta, \;\; X_{33} = -P_2 \sin\theta + P_3 \cos\theta, \;\; Y' + U_{32}X_{33} - (1 + U_{33})X_{32} = 0, \tag{4}$$

and the kinematic dependences following from [3, Eq. (1.6)]

$$y' = U_{32} \cos\theta - (1 + U_{33}) \sin\theta, \qquad z' = U_{32} \sin\theta + (1 + U_{33}) \cos\theta. \tag{5}$$

In system (3)–(5), $\theta(t)$ is the independent angle of rotation of the transverse cross section, $U_{32}(t)$ is the transverse-shear deformation, $U_{33}(t)$ is the deformation of longitudinal tension or compression, $X_{3j}(t)$ is the force vector component in the converted coordinate system, and $Y(t)$ is the bending moment relative to the x_1 axis; the prime denotes the derivative with respect to t. We note that the last equation in (4) expresses the nonlinear dependence between the bending and metric deformations of the rod.

The boundary conditions for system (3)–(5) are written in the form

$$y(0) = z(0) = 0, \qquad \theta(0) = 0, \qquad Y(l) = 0. \tag{6}$$

One condition is set at the point $t = l$ owing to the fact that in formulating the static equations, the value of the force vector (2) known at this point is used.

Without loss of generality, hereinafter, the boundary-value problem (3)–(6) is considered on the unit interval $0 \leqslant t \leqslant 1$. Here it is transformed to the system

$$\begin{gathered} y_0' = y_1, \quad y_1' = f_2 - (\gamma - 1)\varepsilon^2 f_2 f_3, \quad y_2' = \varepsilon^2(\gamma f_2 \cos y_0 - f_3 \sin y_0) - \sin y_0, \\ y_3' = \varepsilon^2(\gamma f_2 \sin y_0 + f_3 \cos y_0) + \cos y_0, \quad f_2 = p_2 \cos y_0 + p_3 \sin y_0, \quad f_3 = -p_2 \sin y_0 + p_3 \cos y_0 \end{gathered} \tag{7}$$

with conditions

$$y_3(0) = y_2(0) = y_0(0) = y_1(1) = 0. \tag{8}$$

The functions $y_0 = \theta$, $y_1 = lY/H$, $y_2 = y/l$, and $y_3 = z/l$ and the parameters $\gamma = E_3/G_2$, $\varepsilon^2 = I_2/(Al^2)$, and $p_j = l^2P_j/H$ are introduced into (7) and (8). System (7) describes the nonlinear elastic bending of the initially straight rod in the plane (x_2, x_3) at given values of the loading, p_2 and p_3, and rigidity, γ and ε, parameters and under specified fixing conditions for the rod. The parameter ε is a small quantity of the order

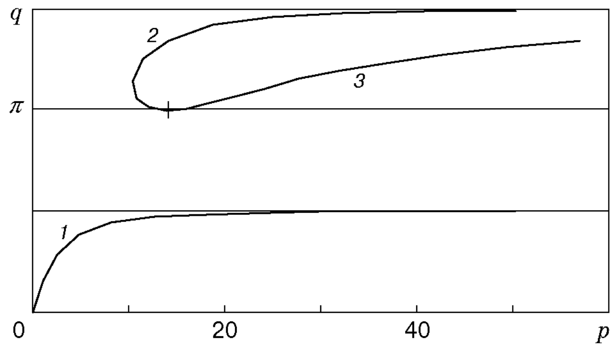

Fig. 1

of the ratio of the rod thickness to its length. For $\varepsilon = 0$, Eqs. (7) degenerate into the classical Euler equations [1, 2], which describe the nonlinear bending of a rod without tension and shear.

Being independent, system (7) admits one quadrature. The first two equations are reduced to one second-order equation relative to the function $y_0 = \theta(t)$: $y_0'' = f_2 - (\gamma - 1)\varepsilon^2 f_2 f_3$, from which follows the quadrature

$$(y_0')^2 = (y_1)^2 = 2 \int\limits_{y_0(1)}^{y_0} [f_2 - (\gamma - 1)\varepsilon^2 f_2 f_3]\, dy_0 \tag{9}$$

taking into account the last condition in (8). Equation (9) can be used for a preliminary analysis of the solutions of problem (7), (8). In particular, it follows from this equation that in the regions where the integral (9) takes on negative values, the solutions are absent. In the regions with positive values of the integral, each solution should have two branches antisymmetric about the variable $y_1 = \theta'(t)$. It follows from the geometrical considerations that the clockwise and counter-clockwise bendings correspond to these branches.

The nonlinear boundary-value problem (7), (8) was solved by the shooting method: the condition $y_1(0) = k$ was added to conditions (8) and a one-parameter family of solutions $\boldsymbol{y}(p,t)$ of system (7) with conditions

$$y_3(0) = y_2(0) = y_0(0) = 0, \quad y_1(0) = k \tag{10}$$

was constructed numerically at a given value of the parameter k. Here $\boldsymbol{y}$ is the vector of desired functions and p is the variable parameter of the external force (p_2 or p_3). The value of the parameter p corresponding to the solution of the boundary-value problem (7), (8) was found iteratively from the condition $y_1(p,1) = 0$. The Cauchy problem (7), (10) was solved by the fourth-order Runge–Kutta method.

The above-described scheme was applied to numerical analysis of four problems of plane bending of a straight rod which simulate various variants of its loading for $\gamma = 2.5$ and $\varepsilon = 0.02$.

Loading by a Translational Transverse Force. The force vector (2) remains collinear to the basis vector $\boldsymbol{e}_2$ and the coordinate axis x_2 during deformation, so that $P_2 = P$ and $P_3 = 0$ (P is the magnitude of the applied force). The functions f_2 and f_3 in system (7) are of the form $f_2 = p\cos y_0$ and $f_3 = -p\sin y_0$, where $p = l^2 P/H$ is the normalized parameter of the external force.

Equation (9) admits the existence of solutions at positive and negative values of the parameter p. The symmetry of the problem allows one to be restricted oneself only to positive values.

Figure 1 shows calculation results obtained for three lower branches of the dependence of the kinematic parameter q on the force parameter p. By definition, we have $q = -\theta(1)$ for branch 1 and $q = \theta(1)$ for branches 2 and 3, so that the parameter q is equal to the angle of rotation of the loaded end of the rod.

Approaching asymptotically a linear dependence as $p \to 0$, branch 1 is a branch of the most lower forms of bending. They are stable and only these forms occur under quasistatic loading of the rod. The straight line $q = \pi/2$ is an asymptotic line of the branch as $p \to +\infty$ (the straight line $q = -\pi/2$ is an asymptotic line as $p \to -\infty$). For two points (p,q) of branch 1, the forms of equilibrium of the rod are shown in Fig. 2 (curve 1 refers to $p = 4.83$ and $q = 1.202$ and curve 2 refers to $p = 50$ and $q = 1.569$). The value of the parameter q of the second form is close to the limit value $\pi/2$.

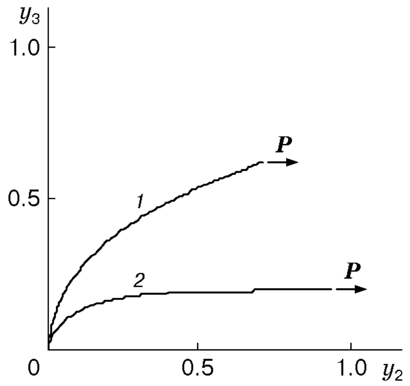

Fig. 2

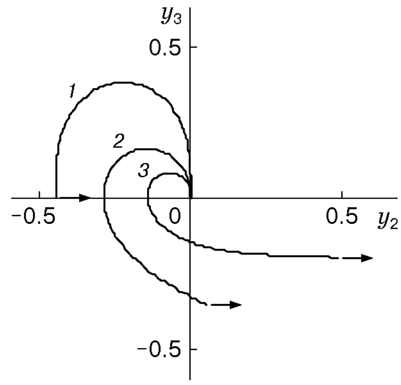

Fig. 3

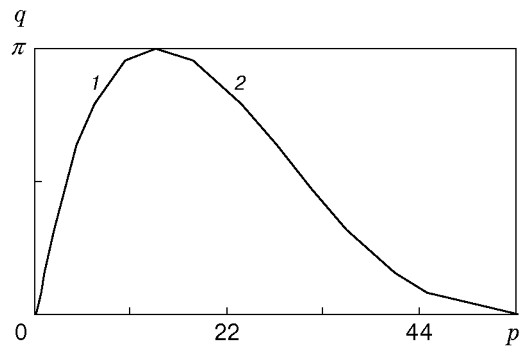

Fig. 4

Branches 2 and 3 in Fig. 1 emanate from the common point ($p \approx 13.75$, $q = \pi$) and they are located on the band $\pi \leqslant q < 3\pi/2$. The forms of bending corresponding to points of branch 2 are shown in Fig. 3, curve 1 corresponding to the common point, curve 2 to the point ($p \approx 14.13$, $q = 4.227$), and curve 3 to the point ($p = 50$, $q = 4.692$).

Loading by a Following Transverse Force. During deformation the force vector (2) is normal to the base line at the boundary point $t = 1$, so that $P_2 = P\cos q$ and $P_3 = -P\sin q$ [$q = -\theta(1)$]. The functions f_2 and f_3 in system (7) have the form $f_2 = p\cos(y_0 + q)$ and $f_3 = -p\sin(y_0 + q)$, where p is the parameter of the external force. Equation (9) admits the existence of solutions at positive and negative values of the parameter p.

Results of the solution of the boundary-value problem (7), (8) are given in Figs. 4–6. Two most lower branches of the dependence $q(p)$ are plotted in Fig. 4. These branches have the common point ($p \approx 13.75$, $q = \pi$), which is the same as in case of loading by a translational transverse force, and they are located in the finite region ($0 \leqslant p < 55$, $0 \leqslant q \leqslant \pi$). The change in the forms of equilibrium of the rod is shown in Figs. 5 and 6 for branches 1 and 2, respectively. Curve 1 in Fig. 5 corresponds to the point ($p = 3.25$, $q = 1.5$), curve 2 to the point ($p = 6.59$, $q = 2.5$), and curve 3 to the point ($p = 13.75$, $q = \pi$). Curve 1 in Fig. 6 corresponds to the point ($p = 23.45$, $q = 2.5$), curve 2 to the point ($p = 31.36$, $q = 1.5$), and curve 3 to the point ($p = 54.6$, $q = 0$).

Loading by a Translational Longitudinal Force. During deformation the force vector (2) remains collinear to the basis vector $\boldsymbol{e}_3$ and the x_3 coordinate axis, so that $P_2 = 0$ and $P_3 = -P$ (the sign is chosen in such a way that the load on the straight rod is compressive for $P > 0$ and stretching for $P < 0$). The functions f_2 and f_3 in system (7) have the forms $f_2 = -p\sin y_0$ and $f_3 = -p\cos y_0$, respectively. In this case, a trivial solution of problem (7), (8) that determines the rectilinear forms of equilibrium of the rod (compressive for $p > 0$ and stretching for $p < 0$)

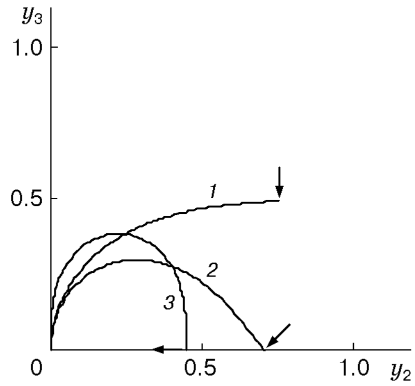

Fig. 5

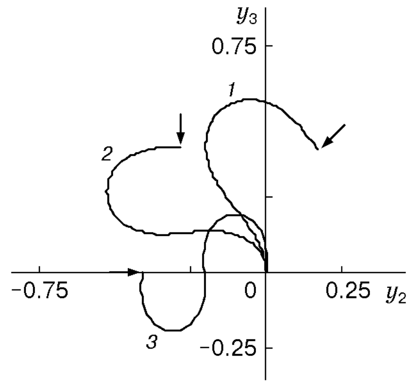

Fig. 6

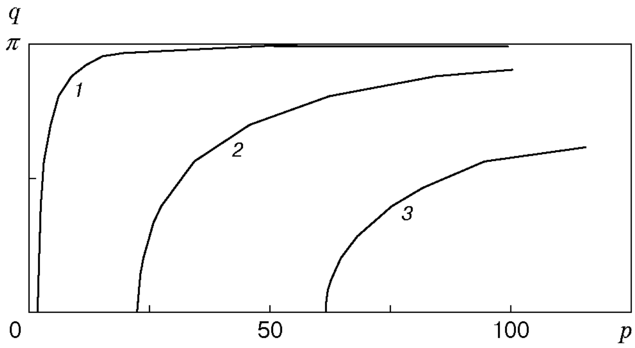

Fig. 7

$$y_0(t) \equiv 0, \qquad y_1(t) \equiv 0, \qquad y_2(t) \equiv 0, \qquad y_3(t) = (1 - \varepsilon^2 p)t \tag{11}$$

exist. Euler showed that the degenerate nonlinear problem ($\varepsilon \to 0$) has a countable set of nontrivial solutions. Later, these solutions were expressed in terms of elliptic Legendre integrals and the results obtained by Euler were supported and complemented [1, 2].

A numerical analysis was performed to find the nontrivial solution branches of the generalized boundary-value problem (7), (8), i.e., the forms different from (11).

Calculation of the integral (9) leads to the equation

$$(y_0')^2 = 2p(\cos y_0 - \cos q) + \frac{\gamma - 1}{2}\,\varepsilon^2 p^2 (\cos 2y_0 - \cos 2q).$$

Together with the rectilinear form $y_0 \equiv q \equiv 0$, this equation admits the bent forms $|y_0| \leqslant |q|$ for positive values of the parameter p.

Figure 7 shows calculation results obtained for three nontrivial branches with the smallest values of p at the branchpoints, i.e., at the points of intersection of the branches with the abscissa (trivial branch). All the branches are located on the band $0 \leqslant q < \pi$ if the kinematic parameter q is determined by the equality $q = |\theta(1)|$. The bent forms of equilibrium of a cantilever rod corresponding to the first two branches are shown in Figs. 8 and 9 for different values of p and q. Curve 1 in Fig. 8 corresponds to the point ($p = 5.03$, $q = 2.202$), curve 2 to the point ($p = 9.34$, $q = 2.758$), and curve 3 to the point ($p = 25.01$, $q = 3.1$). Curve 1 in Fig. 9 corresponds to the point ($p = 22.71$, $q = 0.423$), curve 2 to the point ($p = 27.05$, $q = 1.23$), and curve 3 to the point ($p = 45.33$, $q = 2.2$). If one maps the curves in Figs. 8 and 9 symmetrically about the horizontal axis, they will take the form of Euler elasticas for a rod of length $2l$ which is freely supported at both ends [4]. It is noteworthy that in addition to symmetric forms of bending, a hinged rod also has alternating antisymmetric forms [1].

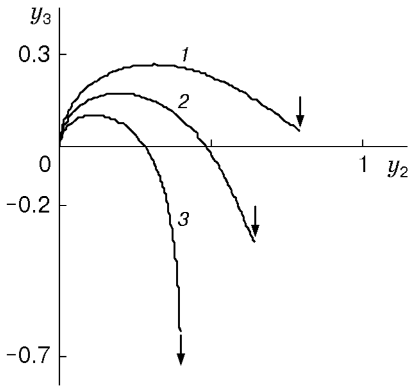

Fig. 8

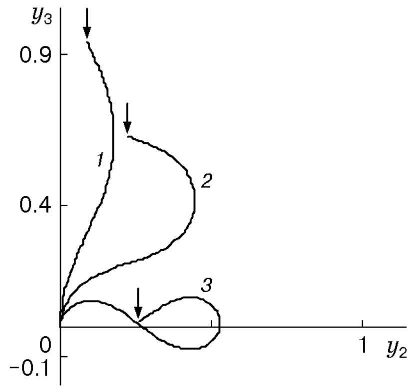

Fig. 9

Only the forms corresponding to the first branch of the dependence $q(p)$ are realized under quasistatic loading (see Fig. 8). However, the higher forms of bending are not only a consequence of the static equations. They occur, for example, upon pulse loading of the rod [5].

Loading by a Following Longitudinal Force. Upon deformation, the force vector (2) is tangent to the base line at the boundary point $t = 1$, so that $P_2 = P \sin q$ and $P_3 = P \cos q$ $[q = -\theta(1)]$. In contrast to the previous problem, the force $P > 0$ is stretching, and the force $P < 0$ is compressive. The functions f_2 and f_3 in system (7) take the forms $f_2 = p \sin(y_0 + q)$ and $f_3 = p \cos(y_0 + q)$, respectively.

Calculation of the integral (9) yields the equation

$$(y_0')^2 = 2p[1 - \cos(y_0 + q)] - \frac{\gamma - 1}{2}\, \varepsilon^2 p^2 [1 - \cos(2y_0 + 2q)], \tag{12}$$

which, obviously, has no solutions for $p < 0$ ($\gamma > 1$ for all real materials). This means that the straight rod has no bent forms of equilibrium for the compressive following force. Therefore, the bifurcation criterion is not suitable for studying the instability of the rectilinear forms of equilibrium of a rod compressed by a following force [6, 7].

For $p > 0$ (stretching following force), Eq. (12) admits the existence of bent forms alongside with a stable rectilinear form $y_0 \equiv q \equiv 0$. These forms are possible at large values of the loading parameters and are of no interest for practice.

Conclusions. Testing of the algorithm for numerical analysis of degenerate problems having the solution in elliptic integrals [1, 2] has shown its high accuracy and efficiency. At the same time, the numerical analysis has confirmed the correctness of the set of Euler equations in the interval $0 \leqslant p \ll O(\varepsilon^{-2})$ determined by an asymptotic analysis of the perturbed system (7).

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